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COMMENT

Fractal sequences derived from the self-similar extensions of the Sierpinski gasket

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Abstract. The well known Sierpinski gasket has been generalised by us into the generalised Pascal-Sierpinski gaskets (GPSG) of orders (K, L), where both K and L are $\geqslant 2$. It has been shown here that families of self-similar sequences can be derived from these extensions of the Sierpinski gasket when $K = 2$ and L is a prime number.

In a recent communication (Lakhtakia *et al* 1987), we have described a bi-indexed family of fractal planar gaskets which are derivable from the Pascal triangle, and named them the generalised Pascal-Sierpinski gaskets (GPSG) of order (K, L), where both K and $L \geqslant 2$. It has been shown that the only members of this family which are strictly self-similar are those known as the Pascal-Sierpinski gaskets (PSG) (Holter *et al* 1986) of orders $(2, L)$ where L is a prime; the remaining GPSG are only self-affine. We have shown that for the PSG of order L prime, a fractal (similarity) dimension d_L given by

$$d_L = \log(1+2+\ldots+L)/\log(L) \tag{1}$$

can be prescribed. To be noted is the fact that the GPSG of order $(2, 2)$ is the usual Sierpinski gasket (Mandelbrot 1983) and that of order $(2, 3)$ is a gasket described by Bhattacharya (1985). In what follows we shall concentrate on the GPSG of orders (2, L prime).

The self-similar GPSG can be described in terms of the spatial convolution operations (Goodman 1968); these gaskets possess a scale factor, also of L. As such, it makes sense for us to truncate the gasket when the number of rows is an integral power of L, i.e. we define levels of evolution $E \geqslant 1$, each level containing the first L^{E+1} rows; the fundamental level is given by $E = 1$. We have also defined a mathematical representation for each level by the function $f_E(x, y; L)$ which can be generated using the relation

$$f_E(x, y; L) = f_{E-1}(x, y; L) * g_E(x, y; L) \tag{2}$$

where $*$ denotes a convolution (Goodman 1968), the array factor $g_E(x, y; L)$ being given as

$$g_E(x, y; L) = \sum_{p=3,5,\ldots}^{L} \sum_{q=2,4,\ldots}^{p-1} [\delta\{x-(p-1)L^E a, y-qL^E b\} + \delta\{x-(p-1)L^E a, y+qL^E b\}]$$
$$+ \sum_{p=2,4,6,\ldots}^{L} \sum_{q=1,3,5,\ldots}^{p-1} [\delta\{x-(p-1)L^E a, y-qL^E b\}$$
$$+ \delta\{x-(p-1)L^E a, y+qL^E b\}]$$
$$+ \sum_{p=1,3,5\ldots}^{L} [\delta\{x-(p-1)L^E a, y\}]. \tag{3}$$

In (3), $\delta\{x-x', y-y'\}$ is the Dirac delta function; while a as well as b are the length parameters of the triangular grid over which the gasket is constructed (Lakhtakia *et al* 1986a). Elsewhere (Holter *et al* 1986) can be found illustrations of some of the resulting structures. What turned out to be quite interesting, however, was that not all attributes of these gaskets are necessarily self-similar: thus, Hilbert curves, drawn in a specific manner on these gaskets, are only self-affine and d_L is merely the asymptotic topological dimension of these curves (Lakhtakia *et al* 1986b).

It is obvious that the gaskets described above are fractals in space, with their spatial (similarity) dimensions lying between log(3)/log(2) and 2, i.e.

$$\log(3)/\log(2) \leq d_L \leq 2 \qquad L \text{ prime} \tag{4}$$

with d_L increasing as L does. But, as has been noted chiefly by Shlesinger (1986), there is no reason why fractals cannot exist in time as well. In fact, we will now show that the GPSG of orders (2, L prime) give rise to fractal sequences which can have importance in time domain problems, such as for data encryption applications or for constructing hard wall diffusors (Schroeder 1986).

For that purpose let $n = 1, 2, 3, \ldots$ be the row number of GPSG, and let $t[n; L]$ be the total number of nodes contained in the first n rows; then $\{t[n; L]\}$ is a sequence obeying the power law

$$t[L \cdot n; L] = (L^{**}d_L) \cdot t[n; L] \tag{5}$$

with ** denoting exponentiation, i.e. $\alpha^{**}\beta = \alpha^\beta$. It should be noted that the sequence $\{t[n; L]\}$ is completely specified by only a few of its members. First of all, $t[1; L]$ and $t[L; L]$ are inter-related as per

$$t[L; L] = t[L \cdot 1; L] = (L^{**}d_L) \cdot t[1; L]. \tag{6a}$$

On noting that $t[1; L] = 1$ and using (1), it can be easily seen that (6a) transforms to

$$t[L; L] = L^{**}d_L = L(L+1)/2. \tag{6b}$$

Further specification of the sequence $\{t[n; L]\}$ also comes from defining $t[m; L]$ for all m which are relatively prime to L; i.e. for $m = 2, 3, 4, \ldots, L-2, L-1$, as well as for all $m > L$ which do not have L as one of their prime divisors. As L increases, an increasingly larger number of such specifications become necessary. Once these specifications have been made, the sequence $\{t[n; L]\}$ is completely determined: this is because for all integers which have L as one of their prime divisors, (5) provides for

$$t[L^q \cdot m; L] = (L^{**}(q \cdot d_L)) \cdot t[m; L] \qquad q = 1, 2, 3, \ldots \tag{7}$$

given that m and L have 1 as their greatest common divisor. In fractal parlance, the numbers $t[m; L]$, such that either $m = 1$ or m and L are relatively prime, are the initiators of the sequence $\{t[n; L]\}$; while the power law (7) serves as its generator.

It is illuminating to give the first few members of some of the sequences so generated. First, for $L=2$, the sequence is given as

$$1, 3, 5, 9, 11, 15, 19, 27, 29, 33, 37, 45, 49, 57, 65, 81, 83, 87, 91, 99, 103, 111, 119, 135, 139, 147, 155, 171, 179, 195, 211, 243, \ldots \qquad (L=2). \tag{8a}$$

It is a simple matter to verify that this sequence satisfies the requirements of a fractal sequence. Its fractal nature is even more transparently seen through the sequence

$$1|2|2, 4|2, 4, 4, 8|2, 4, 4, 8, 4, 8, 8, 16|2, 4, 4, 8, 4, 8, 8, 16, 4, 8, 8, 16, 8, 16, 16, 32|\ldots \qquad (L=2). \tag{8b}$$

The sum of the first n members of the sequence (8b) equals the nth member of the fractal sequence (8a). It should be noted that (8b) is aperiodic; however, it has been partitioned by the separators | in order to bring out its character. The string of characters enclosed between two consecutive separators is precisely twice, and in the same order as, the string of all numbers to the left of the left separator.

Similarly, the fractal sequence for $L=3$ is found to be

$$1, 3, 6, 8, 12, 18, 21, 27, 36, 38, 42, 48, 52, 60, 72, 78, 90, 108, 111, 117, 126, 132, 144, 162, 171, 189, 216, \ldots \qquad (L=3). \tag{9a}$$

Corresponding to (8b), the sequence of successive differences is given as

$$1|2; 3|2, 4, 6; 3, 6, 9|2, 4, 6, 4, 8, 12, 6, 12, 18; 3, 6, 9, 6, 12, 18, 9, 18, 27|\ldots \qquad (L=3). \tag{9b}$$

This time, the use of two separators (| and ;) is necessary to reveal the structure of (9b). Thus between the separators |, the sequence (8b) appears as $|\{\zeta\}; \{\xi\}|$; it should be noted that the sequence $\{\zeta\}$ is twice the string of numbers preceding the left separator |, while the sequence $\{\xi\}$ is three times the same string. Sequences similar to (8a, b) and (9a, b) also exist for the higher primes L as well.

Furthermore, the sequence $\{t[n; L]\}$, for given L, can itself be partitioned into sub-sequences which are strictly self-similar. For any $m \neq 0$ such that $t[m; L] \neq 0$ as well, and the greatest common divisor of m and L equals unity, the sequence $\{s[n; m; L]\}$ given by the relations

$$\begin{aligned} s[n; m; L] &= t[n; L] && \text{if } n = mL^q,\ q = 0, 1, 2, 3, \ldots \\ &= 0 && \text{otherwise} \end{aligned} \tag{10}$$

is itself self-similar with the fractal (similarity) dimension d_L. As examples, consider the sequence

$$1, 3, 0, 9, 0, 0, 0, 27, 0, 0, 0, 0, 0, 0, 0, 81, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 243, \ldots \tag{11}$$

which is derived from (8a) with $m=2$ and $L=2$, and the sequence

$$1, 0, 5, 0, 0, 15, 0, 0, 0, 0, 0, 45, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 135, \ldots \tag{12}$$

which is also derived from (8a) with $m=3$ and $L=2$. It should be noted that the termwise—except for the first term, which is, of course, always unity—addition of (11) and (12) also gives a fractal sequence. Of course, the successive members of the sequences $\{s[n; m; L]\}$ would grow unbounded because $d_L > 1$. For practial utilisation, it may be necessary to terminate the parameter q in (10) at some appropriate q_{max}, and some sort of a normalisation procedure may have to be employed. For example, d_L could be multiplied by some number to ensure that the successive non-zero members of $\{s[n; m; L]\}$ decrease, in which case the similarity dimension of $\{s[n; m; L]\}$ would also be changed; or the numbers $s[n; m; L]$ of the truncated sequence may be used to exponentiate $\exp[2\pi i]$ for possible application in the design of arrays (e.g., Schroeder 1979, 1986).

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